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MORE ON MONADIC LOGIC. PART C: MONADICALLY INTERPRETING IN STABLE UNSUPERSTABLE $\mathcal{T}$ AND THE MONADIC THEORY OF ${}^{\omega}\lambda$

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ABSTRACT

We continue investigating the strength of monadic logic in elementary classes. Mainly we show that all stable unsuperstable theories with finite vocabulary are either among the (easily definable class of) hopelessly complicated or are essentially as complicated as a variant of the tree ${}^{\omega \geq}\lambda$.

The main point here is the second section (see [BSh156]): if $\mathcal{T}$ is (a first order) stable, not superstable theory with finite vocabulary (= set of predicates and function symbols), *then* we can, in monadic logic, interpret in it essentially trees $({}^{\omega \geq}\lambda, <)$ with quantification $(Q^{\mathrm{pd}}f)$ (on pressing down functions). (Note: if $(\mathcal{T}_\infty, \text{2nd}) \leqq (\mathcal{T}, \text{Mon})$, this follows immediately as the class of such trees (up to isomorphism) is definable in second order logic, so the statement follows from 2.6.)

So this is another step in the classification of pairs $(\mathcal{T}, Q)$, $\mathcal{T}$ a first order theory, Q a quantifier. This, of course, raises the question of how complicated is the theory of such trees in $L(Q^{\mathrm{pd}})$; this was dealt with in [Sh205], §1, where erroneously we said that the above interpretation appeared in [BSh156]. We give here a revised form of part of [Sh205], §1.

As for [Sh205], §2, note that conjecture 2.14A (on ultrafilters on ω) was

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disproved in Blass–Shelah [BlSh242], but its aim — doing the interpretation in ZFC — was accomplished in Part B; i.e. [Sh284b] (but not interpreting the second order theory of λ — only after the forcing).

The referee points out another wrong quotation of a nonexisting theorem which was corrected by strengthening 2.1 (omitting a saturation demand).

NOTATION. The letter T serves as a tree and as a first order theory for the latter meaning, so we use here $\mathscr{T}$.

§1. Remarks on [Sh205], §1

CONVENTION. Hence, $1.x$ refers to [Sh205], §1 (or the revised/additional ones here).

On the connection between the $L(Q^{\mathrm{pd}})$-theory of K^{ω}_{tr} (trees with ω levels) and of the trees ${}^{\omega>}\lambda$ (pd stands for pressing down) note:

1.3(5). NOTATION. $K^{\omega}_{\mathrm{tr},\lambda} = \{T \in K^{\omega}_{\mathrm{tr}} : (\forall x \in T)[\,|\mathrm{Suc}^x_T| = \lambda]\}$, $K^{\omega}_{\mathrm{tr,hom}} = \bigcup_\lambda K^{\omega}_{\mathrm{tr},\lambda}$.

1.3B. NOTATION. (a) T_1 is a nice subtree of T_0 (both in K^{ω}_{tr}) *if*

(i) $(T_1, \leqq_{T_1})$ is a submodel of $(T_0, \leqq_{T_0})$.

(ii) for every $x \in T_1$,

$$\mathrm{Suc}_{T_1}(x) = \{y \in T_1 : x < y,\ \neg\,(\exists z \in T_1)[x < z < y]\}$$

is a front of T_0, above x, i.e.

$$(\forall z \in T_0)[x < z \rightarrow (\exists y \in \mathrm{Suc}_{T_1}(x))[z \leqq y \vee y \leqq z]].$$

Every branch ($=$ maximal linearly ordered subset) of T_0 to which x belongs, is not disjoint to $\mathrm{Suc}_{T_1}(x)$.

1.4. CLAIM. (1) The $L(Q^{\mathrm{pd}})$-theories of K^{ω}_{tr} and $\{{}^{\omega>}\lambda : \lambda \geqq \aleph_0$ a cardinal$\}$ (i.e. of $K^{\omega}_{\mathrm{hom}}$) are recursive one in the other.

(2) Being a nice subtree is a transitive relation.

(3) Let $T \in K^{\omega}_{\mathrm{tr}}$.

(a) for every tree T for some nice subtree T_1 of T and λ, $T_1 \cong {}^{\omega>}\lambda$.

(4) Every nice subtree of ${}^{\omega>}\lambda$ is isomorphic to ${}^{\omega>}\lambda$, so every nice subtree of a tree from $K^{\omega}_{\mathrm{tr,hom}}[K^{\omega}_{\mathrm{tr},\lambda}]$ belongs to $K^{\omega}_{\mathrm{tr,hom}}[K^{\omega}_{\mathrm{tr},\lambda}]$.

PROOF. Straightforward: first prove (2), (3), then deduce (1) (by the more elaborate Claim 1.4A below).

In fact we can say how much of the model theory is preserved.

1.4A. CLAIM. There are functions H_1, H_2 from the set of sentences in the $L(Q^{\mathrm{pd}})$-logic for trees to itself, such that:

(1) for every sentence ψ in $L(Q^{\mathrm{pd}})$ (with vocabulary $\{<\}$):

(a) $\psi \in \mathrm{Th}(K^{\omega}_{\mathrm{tr,hom}})$ iff $H_1(\psi) \in \mathrm{Th}(K^{\omega}_{\mathrm{tr}})$,

(b) $\psi \in \mathrm{Th}(K^{\omega}_{\mathrm{tr}})$ iff $H_2(\psi) \in \mathrm{Th}(K^{\omega}_{\mathrm{tr,hom}})$.

(2) H_1, H_2 are recursive, so $\mathrm{Th}_{L(Q^{\mathrm{pd}})}(K^{\omega}_{\mathrm{tr}})$, $\mathrm{Th}_{L(Q^{\mathrm{pd}})}(K^{\omega}_{\mathrm{tr,hom}})$ have the same Turing degree.

(3) (a) For such a sentence ψ and $T \in K^{\omega}_{\mathrm{tr}}$, $T \models H_1(\psi)$ *iff* for some nice subtree T_1 of T and λ, $T_1 \cong {}^{\omega>}\lambda$, ${}^{\omega>}\lambda \models \psi$.

(b) For such a sentence ψ and tree T, $T \models H_2(\psi)$ *iff* for some subtree T_1 of T, $T_1 \models \psi$.

1.4B. CLAIM. (1) Those interpretations do not change the Lowenheim number of sentences, i.e. the H_1, H_2 from 1.4A for $\psi \in L(Q^{\mathrm{pd}})$ satisfy:

$$\mathrm{Min}\{\lambda : {}^{\omega>}\lambda \models \psi\} = \mathrm{Min}\{|T| : T \in K^{\omega}_{\mathrm{tr}}, T \models H_2(\psi)\},$$

$$\mathrm{Min}\{|T| : T \in K^{\omega}_{\mathrm{tr}}, \mathrm{T} \models \psi\} = \mathrm{Min}\{\lambda : {}^{\omega>}\lambda \models H_1(\psi)\}.$$

(2) Hence for the logic $L(Q^{\mathrm{pd}})$, the classes K^{ω}_{tr}, $K^{\omega}_{\mathrm{tr,hom}}$ have the same Lowenheim numbers.

PROOF. Easy.

Note 1.13 can be slightly improved to:

1.13′. LEMMA. (1) *For every sentence ∂ in second order logic we can compute a sentence ∂^* in $L_{\omega,\omega}(Q^{\mathrm{pd}})$ (tree language) such that the following are equivalent*:

(i) $\Vdash_{\mathrm{co}\, l(\aleph_0,\lambda)}$ "$(\alpha, <) \models \partial$ *for some* α, $\alpha < \lambda^{+}$",

(ii) *if* $T \in K^{\omega}_{\mathrm{tr}}$, $|T_{\geq x}| = \lambda$ *for every* $x \in T$, *then* $(T, <) \models \partial^*$.

(2) *If* $V = L$ *we can add*

(iii) $(\lambda, <) \models \partial^*$.

PROOF. By 1.12, 1.8, 1.9.

1.15′. PROOF. The reference is wrong; see Theorem 2.6.

1.18. CONCLUSIONS. (1) If (the first order theory) $\mathscr{T}$ is countable deep and superstable (or just stable), then the Lowenheim number of $L_{\omega_1,\omega}(\mathrm{Mon})$ on $\mathscr{T}$ is $\geqq$ the Lowenheim number of $L_{\omega_1,\omega}(Q^{\mathrm{pd}})$ on $K^{<\omega}_{\mathrm{tr}}$, if $(\mathscr{T}_\infty, \text{2nd}) \not\leqq (\mathscr{T}, \mathrm{Mon})$ then equality holds.

(2) So if $V = L$, it is the Lowenheim number of 2nd order logic.

(3) If $\mathscr{T}$ is stable not superstable, the same conclusions hold.

(4) If $\mathscr{T}$ is uncountable, just replace $L_{\omega_1,\omega}$ by $L_{|T|^+,\omega}$.

PROOF. (1) If $(\mathscr{T}_\infty, \text{2nd}) \leqq (\mathscr{T}, \text{Mon})$ — there are no problems. So assume not. By [BSh156], (Theorem 7.1.14, p. 284; Definition 2.1.3, p. 241) we can interpret $(\mathscr{T}, L_{\omega_1,\omega}(\text{Mon}))$ in $(K_{\text{tr}}^{<\omega}, L_{\omega_1,\omega}(\text{Mon}))$, and *vice versa*, by formulas in $L_{\omega_1,\omega}(\text{Mon})$, hence by 1.17, $(K_{\text{tr}}^{<\omega}, L_{\omega_1,\omega}(Q^{\text{pd}}))$.

(2) By (1) and 1.14.

(3) Similar, using [BSh156], 7.1.14.

[I.e., in addition to the vocabulary of $\mathscr{T}$]:

1.19. CONCLUSION. If $\mathscr{T}$ has finite vocabulary (= language), is stable, not superstable, then

$$\text{Lowenheim Number } (\mathscr{T}, \text{Mon}) \geqq \text{Lowenheim Number } (K_{\text{tr}}^{\omega}, Q^{\text{pd}}).$$

PROOF. By 1.15.

1.20. DISCUSSION. (1) Suppose for simplicity $\mathscr{T}$ has finite language. If $\mathscr{T}$ is superstable $(\mathscr{T}_\infty, \leqq) \not\leqq (\mathscr{T}, \text{Mon})$, (so NDOP) and $\mathscr{T}$ is deep, $\mathscr{T}$ may still have small Lowenheim number $\aleph_0$, e.g. $({}^{\omega>}\lambda, R)$,

$$R = \{(\eta, \nu) : (\exists n)[\lg(\eta) = n + 1 \wedge \nu = \eta \restriction n]\}$$

or large (see 1.16). We are sure there is a nice classification but have not worked on this.

(2) We *know* that for unstable T

$$\text{Hanf Number } (\mathscr{T}, \text{Mon}) = \text{Hanf (2nd order logic)}$$

see [Sh284b], 6.1.

(3) We *think* that the conclusions which assume $V = L$ cannot be proved in ZFC.

§2. On stable $\mathscr{T}$, $(\mathscr{T}_\infty, \text{2nd}) \not\leqq (\mathscr{T}, \text{Mon})$

We assume here some knowledge of [Sh-a], [BSh156].

2.0. HYPOTHESIS. Suppose $\mathscr{T}$ is stable, $(\mathscr{T}_\infty, \text{2nd}) \not\leqq (\mathscr{T}, \text{Mon})$ and the vocabulary of T is finite with relations only. ($\mathfrak{C} = \mathfrak{C}_T$, models will be $\prec \mathfrak{C}$.)

2.0A. NOTATION. $n(\mathscr{T})$ is the maximal arity of atomic formula of $L(T)$.

2.1. LEMMA. *There are formulas* $\varphi(x, y, \bar{z})$, $\psi(y, \bar{z}, \bar{t})$ such that the following holds:

(a) *if* $a \in M$ *and* $M_0 < M$, *then* $\mathrm{tp}_{\varphi(x;y,\bar{z})}(a, M_0)$ *is definable by* $\psi(y, \bar{z}, \bar{d})$ *for some* $\bar{d} \in M_0$,

(b) *define* $R_M^{\varphi,\psi}$ *for* $M < \mathfrak{C}$:
a $R_M^{\varphi,\psi}$ b *iff* $\mathrm{tp}_\omega(a, M \cup \{b\})$ *is not definable by* $\psi(y, \bar{z}, \bar{d})$ *where* $\bar{d}$ *is some (any) sequence from* M *such that* $\mathrm{tp}_\varphi(a, M)$ *is definable by* $\psi(y, \bar{z}, \bar{d})$.

Then

(i) a $R_M^{\varphi,\psi}$ b *implies* a E_M b, $a, b \notin M$,

(ii) *if* $M < N < \mathfrak{C}$, a E_M b, $a \in N$, $b \in N$ then *there are n and* $c_i \in N$ *for* $i = 0$, *n such that* $c_0 = a$, $c_n = b$ *and for* $i < n$ *we have* $c_i R_M^{\varphi,\psi} c_{i+1}$.

REMARK. On E_M see [BSh156], 4.2.1 and Lemma 4.2.6 (p. 259), 4.2.7 (p. 261) and 2.2.4.

PROOF. First we know that if $a_i \in \mathfrak{C}$ for $i < n$, $m < n$, $M < \mathfrak{C}$, and $\mathrm{tp}(\langle a_m, \ldots, a_{n-1}\rangle, M \cup \{a_0, \ldots, a_{m-1}\})$ forks over M, *then* for some $i \in [m, n-1]$, $j \in [0, m]$, $\mathrm{tp}(a_i, M \cup \{a_j\})$ forks over M.

Hence *if* $a_i \in \mathfrak{C}$ for $i < n(\mathscr{T})$, $m < n(\mathscr{T})$,

$$\mathrm{tp}_{\mathrm{qf}}(\langle a_m \cdots a_{n(T)-1}\rangle, M \cup \{\langle a_0, \ldots, a_{m-1}\rangle\})$$

is not definable by any ($\equiv$some) formula over M which defines its restriction to M, *then* for some $i \in [m, n(T))$, $j \in [0, m)$, formula φ, and $\bar{d} \in M$ the type $\mathrm{tp}_{\varphi(x,y,\bar{d})}(a_i, M \cup \{a_j\})$ is not definable by any ($\equiv$some) formula over M which defines its restriction to M.

By compactness (the vocabulary of $\mathscr{T}$ is finite!) there are formulas $\varphi_1, \ldots, \varphi_k$ so that one will be suitable for any $m, M, a_0 \ldots, a_{n(T)-1}$ (note — we may replace the model M by another). By obvious monotonicity and coding (see [Sh-a,II]), we can use one φ. Then by [Sh-a,II] we can find ψ such that (a) holds.

Let us prove (b). Now (i) is easy by the theory of forking (and the definition of E_M). Let us prove (ii), so $M < N < \mathfrak{C}$, $a\, E_M\, b$, $a, b \in N$. Let

$$A = \{c \in N\colon \text{there are } c_0, \ldots, c_n, c_0 = a, c_n = c \text{ and } c_i R_M^{\varphi,\psi} c_{i+1}, \text{ and } c_i \in N\}.$$

Let $B = N \setminus M \setminus A$. By the choice of φ, for any atomic formula $\theta(\bar{x}, \bar{y}, \bar{z})$, $\bar{a} \in A$, $\bar{b} \in B$ ($\lg(\bar{a}) = \lg(\bar{x})$, $\lg(\bar{b}) = \lg(\bar{y})$), we have: $\mathrm{tp}_\theta(\bar{a}, M \cup \bar{b})$ is definable by a formula over M. Now $\mathrm{tp}(A, M \cup B)$ does not fork over M because of the following quite general claim, so once we prove 2.1A, the proof of 2.1 is finished.

2.1A. CLAIM. Suppose N is a model of $\mathscr{T}$, $M \prec N$, $N = A \cup B \cup |M|$, the sets A, B, $|M|$ are pairwise disjoint, then $\mathrm{tp}(A, M \cup B)$ does not fork over M provided that

(*) for any atomic $\theta = \theta(\bar{x}, \bar{y}, \bar{z})$

$(\otimes)_{\theta(x,y,z)}$ if $\bar{a} \in {}^{\lg x}\bar{A}$, $\bar{b} \in {}^{\lg y}B$, $\bar{d} \in M$,

$\varphi(\bar{y}, \bar{z}, \bar{d})$ define the θ-type of $\bar{a}$ over M,

then for $\bar{c} \in {}^{\lg z}|M|$,

$$\theta(\bar{a}, \bar{b}, \bar{c}) \quad \text{if} \quad \varphi(\bar{b}, \bar{c}, \bar{d});$$

equivalently

(**) $\mathrm{tp}_{\mathrm{qf}}(A, |M| \cup B)$ is definable over M.

Also $N \restriction (M \cup A)$, $N \restriction (M \cup B)$ are elementary submodels of M.

PROOF. First observe that without loss of generality, the model (N, M, A, B) is $(2^{|T|})^+$-saturated ((N, M, A, B) is the expansion of N by three unary relations; note that still we do not get the assumption of [Bsh156], 4.3.6 (p. 266) — N is $\| M \|^+$-saturated). This is because by [Sh-a,Ch.II], for any atomic θ, we can choose one $\varphi = \varphi_\theta$ defining any θ-type p over A with parameters from A (for A with $\geqq 2$ elements).

Let us phrase, for a formula $\theta = \theta(\bar{x}, \bar{y}, \bar{z})$, a condition, which is clearly sufficient:

$\otimes_{\theta(x,y,z)}$ *if* $C \subseteq M$, $|C| \leqq |T|$, $\bar{b}_i \in {}^{\lg(y)}M$ for $i < |T|^+$, $\bar{b} = \bar{b}_{|T|^+} \in {}^{\lg(y)}(M \cup B)$, $\{\bar{b}_i : i \leqq |T|^+\}$ is an indiscernible set based on C (satisfaction in $\mathfrak{C}$, $M \prec N \prec \mathfrak{C}$), $\bar{a} \in {}^{\lg x}(M \cup A)$, $\mathrm{tp}(\bar{a}, M)$ does not fork over C, and $\bar{c} \in C$ then:

$$\vDash \theta[\bar{a}, \bar{b}, \bar{c}] \text{ iff } \vDash \theta[\bar{a}, \bar{b}_i, \bar{c}] \text{ for every } (\equiv \text{some}) \ i < |T|^+.$$

We prove this by induction on θ: first on the quantifier depth of θ and then on length (simultaneously for all M, N, A, B, C, $\bar{a}$, $\bar{c}$, $\langle b_i : i \leqq |T|^+ \rangle$ as above).

For θ atomic — this follows by (*).

For θ being $\neg\, \theta_1$ or $\theta_1 \wedge \theta_2$ — trivial by the induction hypothesis.

So assume $\theta(\bar{x}, \bar{y}, z) = (\exists\, w)\theta_1(\bar{x}, w, \bar{y}, \bar{z})$; so let C, $\bar{b}_i$ $(i \leqq |T|^+)$, $\bar{b} = b_{|T|^+}$, $\bar{c} \in C$, $\bar{c} \in {}^{\lg x}(M \cup A)$ be as in the assumption of $\oplus_{\theta(x,y,z)}$. Without loss of generality $C \prec M$ (as we can delete, not too many, b_i's for $i < |T|^+$, and can increase C). We have to prove the "iff" in the conclusion of $\oplus$.

PROOF OF THE "ONLY IF" PART (i. e. $\Rightarrow$). So we assume $\models \theta[\bar{a}, \bar{b}, \bar{c}]$, hence for some $d \in N$, $\models \theta_1[\bar{a}, d, \bar{b}, \bar{c}]$.

First case in the "only if" part: $d \in M$.

Without loss of generality $d \in C$, and apply the induction hypothesis (with d "joining" $\bar{c}$), i.e. applied to $\bar{a}$, $\bar{b}_i$, $\bar{c}\,\hat{}\,\langle d \rangle$.

Second case in the "only if" part: $d \in A$.

Without loss of generality $\mathrm{tp}(\bar{a}\,\hat{}\,\langle d \rangle, M)$ does not fork over C; now apply the induction hypothesis (with d "joining" $\bar{a}$), i.e. applied to $\bar{a}\,\hat{}\,\langle d \rangle$, $\bar{b}_i$, $\bar{c}$.

Third case in the "only if" part: $d \in B$.

Without loss of generality $\mathrm{tp}(\langle d \rangle\,\hat{}\,\bar{b}, M)$ does not fork over C, so we can let $d_{|T|^+} = d$ and choose $d_i \in M$ for $i < |T|^+$ such that $\{\langle d_i \rangle\,\hat{}\,\bar{b}_i : i < |T|^+\}$ is an indiscernible set over C based on $\mathrm{stp}(\langle d \rangle\,\hat{}\,\bar{b}, C)$. We can do it as M is $|T|^+$-saturated. Now apply the induction hypothesis to $\bar{a}$, $\langle d_i \rangle\,\hat{}\,\bar{b}_i$, $\langle d \rangle\,\hat{}\,\bar{b}$, $\bar{c}$.

PROOF OF THE "IF" PART (i.e. $\Leftarrow$). So we assume $\models \theta[\bar{a}, \bar{b}_i, \bar{c}]$ for $i < |T|^+$. So for each $i < |T|^+$, there is $d_i \in N$ such that $\models \theta_1[\bar{a}, d_i, \bar{b}_i, \bar{c}]$.

By [BSh156], 4.2.7 for each i, $\mathrm{tp}(\bar{a}\,\hat{}\,\langle d_i \rangle, C \cup \bar{b}_i)$ does not fork over C or $\mathrm{tp}(\bar{a}, C \cup \bar{b}_i \cup \{d_i\})$ does not fork over C. If the former case occurs, then $j < |T|^+$ & $\mathrm{tp}(\bar{a}\,\hat{}\,\langle d_i \rangle, C \cup \bar{b}_j)$ does not fork over $C \Rightarrow \theta_1[\bar{a}, d_i, \bar{b}_j, \bar{c}]$†; but this occurs for all j large enough, so without loss of generality $d_j = d^*$ for all $j < |T|^+$ for a fixed $d^* \in N$. If the former fails, then necessarily the latter occurs, without loss of generality, $d_i \in M$; and as M is $|T|^+$-saturated without loss of generality, $\{\langle d_i \rangle\,\hat{}\,\bar{b}_i : i < |T|^+\}$ is an indiscernible set over C based on C; also $\mathrm{tp}(d_i, C \cup \bar{b}_i)$ forks over C (by [BSh156], 4.2.7). So the following two cases cover all possibilities.

First Case: d^* well defined.

First Subcase: $d^* \in M$.

Without loss of generality $d^* \in C$ and apply the induction hypothesis to $\bar{a}$, b_i, $\bar{c}\,\hat{}\,\langle d^* \rangle$.

Second Subcase: $d^* \in A$.

Without loss of generality $\mathrm{tp}(\bar{a}\,\hat{}\,\langle d^* \rangle, M)$ does not fork over C; now apply the induction hypothesis (on $\bar{a}\,\hat{}\,\langle d^* \rangle$, $\bar{b}_i$, $\bar{c}$).

† (Remember $C \prec N$, hence $\mathrm{tp}(\bar{a}\,\hat{}\,\langle d_i \rangle, C)$ is stationary.)

Third Subcase: $d^* \in B$.

We will discard this case.

For $n < \omega$ let $\theta_2^n = \theta_2^n(\bar{x}; \bar{w}; \bar{y}_1, \dots, \bar{y}_n, \bar{z}) = \bigwedge_{l=1}^n \theta_1^n(\bar{x}, w, \bar{y}_l, \bar{z})$; it has quantifier depth smaller than that of θ, so applying the induction hypothesis to it we can find $d^{**} \in N$ such that $\models \theta_2^n[\bar{a}, d^{**}, \bar{b}_1, \dots, \bar{b}_n, \bar{c}]$ (let $C \cup \bar{b}_1 \cup \cdots \cup \bar{b}_n \subseteq C' \subseteq M$, $|C'| \leqq |T|$, and $\{d_i^* : i < |T|^+\} \subseteq M$ indiscernible set on C' based on C', so $C', \bar{a}, d_i^*, d^*, \bar{b}_1 {}^\wedge \cdots {}^\wedge \bar{b}_n {}^\wedge \bar{c}$ here play the role of $C, \bar{a}, \bar{b}_i, \bar{b}, \bar{c}$ in $\oplus$). For $n < \omega$ large enough, by [Sh-a,II], we have: for every $i < |T|^+$ large enough, $\models \theta_1[\bar{a}, d^{**}, \bar{b}_i, \bar{c}]$; so we can use the first subcase.

Second Case: $d_i \in M$ for $i < |T|$, $\{\langle d_i \rangle {}^\wedge \bar{b}_i : i < |T|^+\}$ an indiscernible set on C based on C and $\mathrm{tp}(d_i, C \cup \bar{b}_i)$ forks over C.

Choose $d = d_{|T|^+} \in N$ such that $\{\langle d_i \rangle {}^\wedge \bar{b}_i : i \leqq |T|^+\}$ is an indiscernible set on C based on C.

First Subcase: $d \in M$.

Impossible as $\mathrm{tp}(d, C \cup \bar{b})$ forks, over C whereas $\mathrm{tp}(\bar{b}, M)$ does not fork over C.

Second Subcase: $d \in A$.

This case should be impossible, or at least avoidable, as A, B are quite independent over M, whereas $\mathrm{tp}(d, C \cup \bar{b})$ forks over C (part of the assumption of the second case). However, we have not yet proved $\mathrm{tp}(A, M \cup B)$ does not fork over M, only for formulas of quantifier depth smaller than that of θ; but this suffices.

More formally, when we choose the $d_i \in M$ for $i < |T|^+$, without loss of generality there are $\bar{a}_\zeta \in {}^{\lg(\bar{x})}C$ for $\zeta < \omega$, $\{\bar{a}_\zeta : \zeta < \omega\}$ indiscernible, $\mathrm{Av}(\{\bar{a}_\zeta : \zeta < \omega\}, M) = \mathrm{tp}(\bar{a}, M)$. So we know that $(\exists^\infty n)\theta_1(\bar{a}_n, d_i, \bar{b}_i, \bar{c})$ holds (and this is just a finite Boolean combination of $\{\theta_1(\bar{a}_n, d_i, \bar{b}_i, \bar{c}) : n < \omega\}$) and this suffices (for the satisfaction of $\theta_1(\bar{a}, d_i, \bar{b}_i, \bar{c})$) if $d_i \in M$. So if the formula $(\exists^\infty n)\theta_1(\bar{a}_n, y, \bar{b}_i, \bar{c})$ does not fork over C, without loss of generality $\mathrm{tp}(d_i, C \cup b_i)$ does not fork over C, but then we would have fallen to the First Case. So for some (equivalently, every) $i < |T|^+$ the formula $(\exists^\infty n)\theta_1(\bar{a}_n, y, \bar{b}_i, \bar{c})$, hence the formula $(\exists^\infty n)\theta_1(a_n, y, \bar{b}, \bar{c})$, forks over C, hence it is not realized in M. But it is realized in $M \cup B$. But (increasing C) without loss of generality $\mathrm{tp}(d, M)$ does not fork over C, and there is $\{d_i : i < |T|^+\} \subseteq M$ indiscernible over C based on C; d_i realizes $\mathrm{tp}(d, C)$. Now apply the induction hypothesis (interchanging A and B).

Third Subcase: $d \in B$.

As $C \prec M \prec N \leqq \mathfrak{C}$, $\mathrm{tp}(\bar{b}, M)$ does not fork over C, and $\mathrm{tp}(d, C \cup \bar{b})$ does fork over C; (by [Bsh156], 4.2.7) we know $\mathrm{tp}(\langle d\rangle{}^\wedge\bar{b}, M)$ does not fork over C. So we can apply the induction hypothesis (to $\bar{a}$, $\langle d_i\rangle{}^\wedge b_i$, $\langle d\rangle{}^\wedge\bar{b}$, $\bar{c}$).

2.2. Remark. Comparing this with the proof in [Bsh156], 4.3.11, p. 268, the definition for E_A^{for} suggesting itself by the lemma is in some sense more simply defined — we quantify over finite sets (looking for a possible path $c_0, \ldots, c_n$) rather than over formally good sets!

Also in [Bsh156], 4.3.11 we assume "N is $\|M\|^+$-saturated" which is not needed here. In short

2.2A. Claim. If $M < N(<\mathfrak{C})$, $E_M \restriction N$ is definable in the model (N, M) (i.e. N expanded by a unary relation) by a formula in weak monadic logic,[†] and by the (same) formula in monadic logic where:

2.2B. Definition. Weak monadic logic has the same syntax as monadic logic, but in the definition of the satisfaction relation, the monadic variables range over *finite* sets of elements.

Proof of 2.2A. Let φ, ψ be from Lemma 2.1. First note that $xR_M^{\varphi,\psi}y$ is first order definable in (N, M) (use [Sh-a,II]).

Now for both logics, for $a, b \in N$, $a\, E_M\, b$ *iff* for some Z, $a \in Z$, $b \in Z$ and

$$(\forall Y \subseteq Z)[(\forall x, y \in Z)[xR_M^{\varphi,\psi}y \rightarrow (x \in Y \equiv y \in Y)] \Rightarrow a \in Y \equiv b \in Y].$$

If $a\, E_M\, b$ let $Z = \{c_0, \ldots, c_n\}$ from (ii) of 2.1; it satisfies the right side by 2.1(b)(ii).

If $\neg\, a\, E_M\, b$ and Z exemplify the right side, choose $Y = (a/E_M) \cap Z$.

2.3. Claim. If $M < \mathfrak{C}$, φ, ψ are as in Lemma 2.1, then for each n there are finite sets Δ_n^1, Δ_n^2 of formulas such that:

if $c_i R_M^{\varphi,\psi} c_{i+1}$ for $i = 0, 1, \ldots, n-1$, *then*

$$\mathrm{tp}_{\Delta_n^1}(c_0, M \cup \{c_n\}) \text{ forks over } M$$

and

$$\mathrm{tp}_{\Delta_n^2}(c_n, M \cup \{c_0\}) \text{ forks over } M.$$

Proof. Easy; again use compactness.

[†] If the signature of $\mathcal{T}$ is infinite, say of power λ (but $\mathcal{T}$ is stable, $(\mathcal{T}_\infty, \text{2nd}) \not\leq (\mathcal{T}, \text{Mon})$), then 2.1A, 2.2A and 2.5 still hold for the logic $L_{\lambda^+,\omega}$ (which is stronger than weak monadic logic).

2.3A. Remark. Would one Δ suffice for all n? In general, no! For example, let $\mathscr{T}$ be the theory of graphs without cycle. In this case, $x E_M y$ iff x, y are in the same component and if this component C is not disjoint to M, all $z \in C \cap M$ have the same distance from x and from y.

2.4. Definition. For λ let O^*_λ be the model with universe ${}^\omega\lambda \times \omega$ and relations

$$E^* = \{\langle(\eta, n), (\nu, m)\rangle; \eta = \nu\},$$

$$E = \{\langle(\eta, n), (\nu, m)\rangle : n = m, \eta \restriction n = \nu \restriction m\},$$

$$\leqq^* = \{\langle(\eta, n), (\nu, m)\rangle : \eta = \nu, n \leqq m\}.$$

2.5. Theorem. *Suppose $\mathscr{T}$ is stable not superstable with finite vocabulary. Then for every λ for some model N of $\mathscr{T}$ of cardinality $\lambda^{\aleph_0}$, in some expansion N^* of N of three unary predicates, a model isomorphic to O^*_λ is monadically definable in N^* with elements represented by elements.*

Proof. So there are $M_n < M_{n+1}$ and $a \in C$ such that $\mathrm{tp}(a, M_{n+1})$ forks over M_n for each n. Without loss of generality $\| M_n \| \leqq \aleph_0$, and

$$[\bigcup_{l<\omega} M_l \cup \{a\} \setminus M_n]$$

is included in one E_{M_n}-equivalence class. Let $M_\omega = \bigcup_{n<\omega} M_n$, $M_\omega \cup \{a\} \subseteq M_{\omega+1}$, $M_{\omega+1} \setminus M_n \subseteq a/E_{M_n}$, $M_{\omega+1} \setminus M_\omega \subseteq a/E_{M_\omega}$. Let $A = M_{\omega+1} \setminus M_\omega$.

2.5A. Observation. For no n is $\mathrm{tp}_{\mathrm{qf}}(A, M_\omega)$ definable over M_n.
[If so we get by 2.1A that $\mathrm{tp}(A, M_\omega)$ does not fork over M_n, hence $\mathrm{tp}(a, M_\omega)$ does not fork over M_n, contradiction.]

So for each n, for some $m < n(\mathscr{T})$, $a_0, \ldots, a_{m-1} \in M_\omega \setminus M_n$, $a_m \cdots a_{n(\mathscr{T})-1} \in A$, we have $\mathrm{tp}_{\mathrm{qf}}[\langle a_m, \ldots, a_{n(\mathscr{T})-1}\rangle, M_n \cup \{a_0 \cdots a_{m-1}\}]$ forks over M_n. So by the choice of φ (from 2.1), for some $i \in [0, m)$ and $j \in [m, n(\mathscr{T}))$, we have $\mathrm{tp}_{\varphi(x,y,\bar{a})}(a_j, M_n \cup \{a_i\})$ forks over M_n. Then $p_b \overset{\mathrm{def}}{=} \mathrm{tp}_{\varphi(x,y,z)}(b, M_\omega)$ forks over M_n for some $b \in A$. On the other hand, for some $k < \omega$, p_b is definable over some M_k, hence does not fork over some M_k. As we can replace $\langle M_l : l < \omega\rangle$ by any infinite subsequence, without loss of generality we can find $b_n \in A$ such that:

(*) $p_{b_n} = \mathrm{tp}_{\varphi(x,y,z)}(b_n, M_\omega)$ is definable over M_{n+1} but not over M_n.

Let λ be a cardinal. Let, for $\eta \in {}^{\omega>}\lambda$, f_η be an elementary mapping $\mathrm{Dom}\, f_\eta = M_{\lg(\eta)}$, $M_\eta \overset{\mathrm{def}}{=} \mathrm{Rang}\, f_\eta$ such that $\langle M_\eta : \eta \in {}^{\omega>}\lambda\rangle$ is a non-forking tree. Let

$M^* = \bigcup \{M_\eta : \eta \in {}^{\omega>}\lambda\}$; it is of course $\prec \mathfrak{C}$ (as $(T_\infty, \text{2nd}) \not\leq (T, \text{Mon})$). Let, for $\eta \in {}^\omega\lambda$, f_η be an elementary mapping with domain $M_{\omega+1}$ extending $\bigcup_{k<\omega} f_{\eta \restriction k}$ and letting $M_\eta = f_\eta(M_{\omega+1})$; $\langle M_\eta : \eta \in {}^\omega\lambda \rangle$ is independent over M^*. Let $N^* = \bigcup_{\eta \in {}^\omega\lambda} M_\eta \prec \mathfrak{C}$. Note that $E^{\text{for}}_{M^*} \restriction (N^x \setminus M^x)$ has the $A_\eta = M_\eta \setminus M^*(\eta \in {}^\omega\lambda)$ as equivalence classes (remember: if $b, c \in M_{\omega+1} \setminus M_\omega$, then $\text{tp}(b, M_\omega \cup \{c\})$ forks over M_ω — by the choice of $M_{\omega+1}$). Let $a_\eta = f_\eta(a)$, $b^n_\eta = f_\eta(b_n)$ for $\eta \in {}^\omega\lambda$. Easily the expected facts are preserved. So

$$\text{tp}_\varphi(b^n_\eta, M^*) = \text{tp}_\varphi(b^n_\nu, M^*) \qquad \text{iff } \eta \restriction (n+1) = \nu \restriction (n+1).$$

Let $P^a = \{a_\eta : \eta \in {}^\omega\lambda\}$, $P_n = \{b^n_\eta : \eta \in {}^\omega\lambda\}$, $P = \bigcup_{n<\omega} P_n$ and

$$E^* = \{\langle b^n_\eta, b^m_\nu \rangle : \eta = \nu, n < \omega, m < \omega\}.$$

Now in N^*, expanded by monadic predicates for M^*, P, the following are (monadically) definable:

(a) $E^* = \{\langle b^n_\eta, b^m_\nu \rangle : \eta = \nu, m < \omega, n < \omega\}$
because E_{M^*} is monadically definable by 2.2A and A_η $(\eta \in {}^\omega\lambda)$ are the equivalence classes of $E_{M^*} \restriction (N^* - M^*)$ (see above) and
$\eta = \nu \Leftrightarrow b^n_\eta E_{M^*} b^m_\nu$.

(b) $E = \{\langle b^n_\eta, b^m_\nu \rangle : n = m, \eta \restriction (n+1) = \nu \restriction (+1)\}$
say that their φ-types over M^* are equal [if $n = m$ see above; if $n \neq m$, without loss of generality $m < n$, $\text{tp}_\varphi(b^n_\eta, M^*)$ is definable over $M^*_{\eta \restriction (n+1)}$ but not over $M^*_{\eta \restriction n}$ hence (for any $\rho \in {}^{\omega>}\lambda$) not over M^*_ρ, if $\eta \restriction (n+1) \not\leq \rho$ so that negative result follows].

(c) $\leqq^* = \{\langle b^n_\eta, b^m_\nu \rangle : \eta = \nu, n \leqq m\}$
$x \leqq^* y$ iff $xE^*y \wedge (\forall z \in P)[(\exists t)(tE^*z \wedge tEy) \rightarrow (\exists t)(tE^*z \wedge tEx)]$.

(d) $\{\langle a_\eta, b^n_\eta \rangle : n < \omega \text{ and } \eta \in {}^\omega\lambda\}$
same proof as (a).

2.6. Lemma. *If $\mathcal{T}$ is stable not superstable, we can semantically interpret $(K^{\omega+1}_{\text{tr}}, Q^{\text{pd}})$ in $(\mathcal{T}, \text{Mon})$ by monadic formulas (so the information for Lowenheim numbers is preserved).*

Proof. Just manipulate the O^*_λ.

Remark. Remember we are assuming 2.0.

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